Exercise 31

(a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

(c) Show that your answers to parts (a) and (b) are equivalent.

Solution

Use the quotient rule to differentiate $f(x)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[f(x)] \\
&= \frac{d}{dx}\left(\frac{\tan x - 1}{\sec x}\right) \\
&= \frac{\left[\frac{d}{dx}(\tan x - 1)\right]\sec x - \left[\frac{d}{dx}(\sec x)\right](\tan x - 1)}{(\sec x)^2} \\
&= \frac{(\sec^2 x)\sec x - (\sec x \tan x)(\tan x - 1)}{\sec^2 x} \\
&= \frac{\sec^3 x - \sec x \tan^2 x + \sec x \tan x}{\sec^2 x} \\
&= \frac{\sec^2 x - \tan^2 x + \tan x}{\sec x} \\
&= \frac{(\tan^2 x + 1) - \tan^2 x + \tan x}{\sec x} \\
&= \frac{1 + \tan x}{\sec x} \\
&= \frac{1 + \frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \cdot \frac{\cos x}{\cos x} \\
&= \cos x + \sin x
\end{aligned}$$

Write $f(x)$ in terms of sine and cosine

$$f(x) = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}} \cdot \frac{\cos x}{\cos x} = \sin x - \cos x$$

and then differentiate it.

$$f'(x) = \frac{d}{dx}[f(x)] = \frac{d}{dx}(\sin x - \cos x) = \frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x) = (\cos x) - (-\sin x) = \cos x + \sin x$$